Leader Notes: A Golden Idea

Explore/Explain Cycle II

Purpose:

Use a geometric context to generate data that can be modeled with an exponential function. Use technology to develop and analyze the exponential function.

Descriptor:

Participants will use Geometer's Sketchpad to examine a construction of a regular pentagon and a sequence of golden triangles. They will use angle relationships found in the pentagon and triangles created by its diagonals in order to make conjectures about similar triangles. Participants will use Geometer's Sketchpad to measure the lengths of the legs of successive isosceles triangles created by bisecting base angles. This data will be used to generate an exponential decay function. Participants will then dilate a golden isosceles triangle by a scale factor equal to the golden ratio. By measuring leg lengths of successive triangles, participants will gather data that will be used to generate an exponential growth function.

Duration:

2.5 hours

TEKS:

a(5) Tools for algebraic thinking. Techniques for working with functions and equations are essential in understanding underlying relationships. Students use a variety of representations (concrete, pictorial, numerical, symbolic, graphical, and verbal), tools, and technology (including, but not limited to, calculators with graphing capabilities, data collection devices, and computers) to model mathematical situations to solve meaningful problems.

a(6) Underlying mathematical processes. Many processes underlie all content areas in mathematics. As they do mathematics, students continually use problem-solving, language and communication, and reasoning (justification and proof) to make connections within and outside mathematics. Students also use multiple representations, technology, applications and modeling, and numerical fluency in problem-solving contexts.

2A.1(B) collect and organize data, make and interpret scatterplots, fit the graph of a function to the data, interpret the results, and proceed to model, predict, and make decisions and critical judgments.

2A.4(A) Identify and sketch graphs of parent functions, including linear ($f(x) = x$), quadratic ($f(x) = x^2$), exponential ($f(x) = a^x$), and logarithmic ($f(x) = \log_a x$)

functions, absolute value of x ($f(x) = |x|$), square root of x ($f(x) = \sqrt{x}$), and reciprocal of x ($f(x) = \frac{1}{x}$).

2A.4(B) Extend parent functions with parameters such as a in $f(x) = \frac{a}{x}$ and describe the effects of the parameter changes on the graph of parent functions.

2A.11(B) use the parent functions to investigate, describe, and predict the effects of parameter changes on the graphs of exponential and logarithmic functions, describe limitations on the domains and ranges, and examine asymptotic behavior.

2A.11(F) analyze a situation modeled by an exponential function, formulate an equation or inequality, and solve the problem.

G.5(A) Use numeric and geometric patterns to develop algebraic expressions representing geometric properties.

G.10(A) Use congruence transformations to make conjectures and justify properties of geometric figures including figures represented on a coordinate plane.

G.11(A) Use and extend similarity properties and transformations to explore and justify conjectures about geometric figures.

TAKS Objectives Supported by these Algebra 2 and Geometry TEKS:

- Objective 1: Functional Relationships
- Objective 2: Properties and Attributes of Functions
- Objective 6: Geometric Relationships and Spatial Reasoning
- Objective 8: Measurement and Similarity
- Objective 10: Mathematical Processes and Mathematical Tools

Technology:

- Internet access
- Graphing calculator
- Dynamic geometry software (such as Geometer's Sketchpad)
- Spreadsheet (such as Excel)

Materials:

Advanced Preparation: The Geometer's Sketchpad sketch **Golden Triangles.gsp** will need to be installed on each computer for participant use.

Presenter Materials: projector (computer or overhead) for graphing calculator

Per group: Geometer's Sketchpad sketch **Golden Triangles.gsp**

Per participant: graphing calculator, activity sheets

Leader Notes:

The golden ratio has been used in Western culture since the ancient Greeks used it to build the Parthenon and the Egyptians to build the Pyramids at Giza. It permeates Western art and architecture. The golden ratio is found in nature, including proportions in the human and other mammalian bodies, spiral shells, and insects.

Since the golden ratio is so prevalent, it carries with it algebraic implications as well. In this phase of the professional development, participants will use the golden ratio to collect data that can be modeled using an exponential function. They will use transformations to fit a function rule then use that function rule to make predictions.

Explore

Posing the Problem:

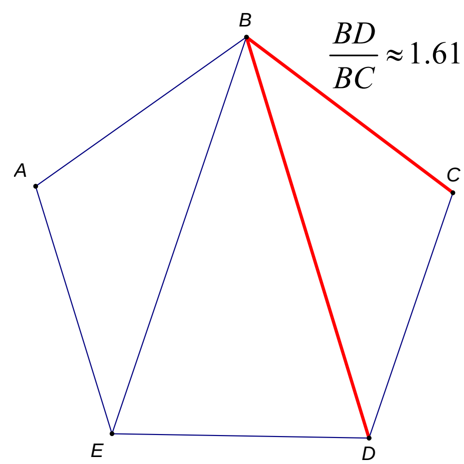
Mathematics has been an important influence throughout civilization, from ancient times to the present. In ancient Egypt, Greece, and Rome, geometry and proportion were used in art and architecture. Medieval Europeans carried this tradition of using proportion as they built beautiful cathedrals. Renaissance painters and sculptors used proportion to convey their idea of natural beauty.

Where does this proportion originate? The Greeks found a certain ratio to be prevalent in the natural world around them.

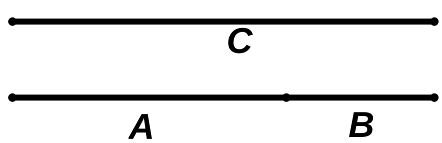
The golden ratio can be developed from several constructions. One way to construct the golden ratio is using the diagonals of a regular pentagon.

In regular pentagon *ABCDE* (shown at right), the ratio of the length of a diagonal from vertex B to the length of a side of the pentagon is always the same. This ratio is called the **golden ratio**, which the Greeks notated with the capital letter *phi*, or Φ.

$$\frac{BD}{BC} \approx 1.618 = \Phi$$



From a segment length perspective, the golden ratio is a geometric mean. Geometrically, segment (in the diagram below, of length *C*) can be split into two smaller segments (of lengths *A* and *B*).

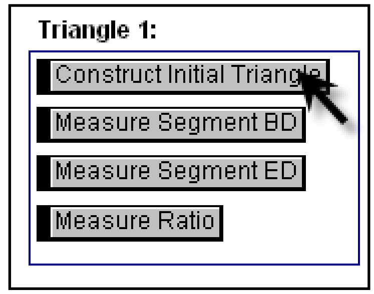


The splitting of the segment is such that the ratio of the length of the original segment to the length of the larger piece (C:A) is the same as the ratio of the length of the larger piece to the length of the smaller piece (A:B). In other words,

$$\frac{C}{A}=\frac{A}{B}$$

If the golden ratio is applied in succession to a geometric construction, what types of functional behaviors are present?

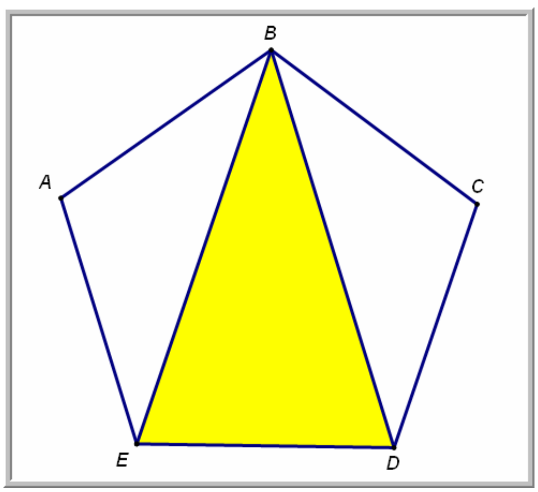
Part 1: Investigating Leg Length



Open the Geometer's Sketchpad sketch "Golden Triangles.GSP." If necessary, click on the "Investigating Leg Length" Tab. Pentagon *ABCDE* is a regular pentagon. From this regular pentagon, a series of triangles can be constructed. Click on the "Construct Initial Triangle" action button.

1. What kind of triangle is ΔBED? How do you know?

ΔBED is an isosceles triangle. Because ABCDE is a regular pentagon, $\overline{AB}\cong\overline{CB}$ and $\overline{AE}\cong\overline{CD}$. also, $m\angle A=108°$ and $m\angle C=108°$, so $\angle A\cong\angle C$. By SAS triangle congruence, $\Delta ABE\cong\Delta CBD$. Since corresponding parts of congruent triangles are congruent, $\overline{BE}\cong\overline{BD}$ and ΔBED is isosceles.



Facilitation Questions

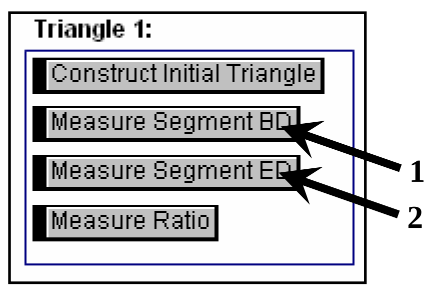
- Do you see any congruent sides? How do you know they're congruent?
  Yes; all five sides of the pentagon are congruent because of the definition of a regular pentagon.
- What are the angle measures of the pentagon?
  In a regular pentagon, the interior angles all measure 108°.
- What is the sum of the measures of the interior angles of any triangle?
  180°
- What are the angle measures of the interior triangles?
  For ΔABE and ΔBCD the interior angles are 108°, 36°, and 36°. For ΔBED, the interior angles are 72°, 72°, and 36°.
- How could you use congruent triangles to show that some segments are congruent?
  Since corresponding parts of congruent triangles are congruent, if we can show that $\Delta ABE \cong \Delta CBD$, then we can show that $\overline{BE} \cong \overline{BD}$.

2. **Is your triangle classification from Question 1 true for every triangle formed when two diagonals are drawn from one vertex of a regular pentagon? How do you know?**
   Yes. For any regular pentagon, the diagonals from the same vertex will be congruent regardless of their length. Thus, the interior triangle will always be isosceles. (Incidentally, the other two triangles, in this case ABE and CBD, are also isosceles.)

Measure the length of $\overline{BD}$ by clicking on the "Measure Segment BD" action button.
Measure the length of $\overline{ED}$ by clicking on the "Measure Segment ED" action button.

3. **What is the ratio of the length of $\overline{BD}$ to the length of $\overline{ED}$? How did you find this ratio?**
   1.618, which can be found be either dividing BD by ED or by clicking on the Measure Ratio action button. If participants are fluent with The Geometer's Sketchpad, they may also be able to use the software to calculate the ratio.

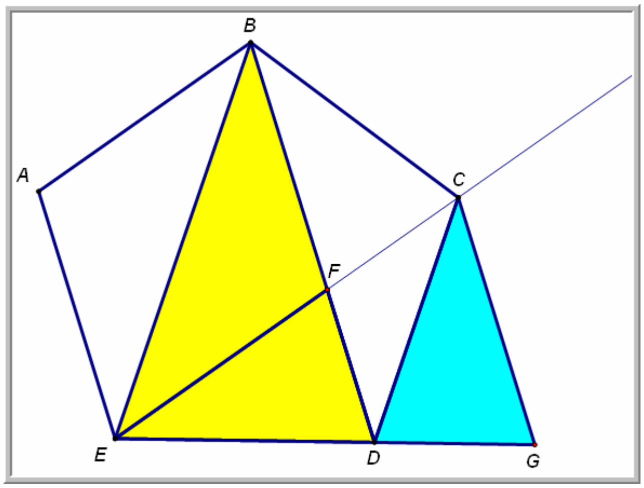
4. **What does this ratio represent?**
   1.618 is the golden ratio, Φ



Click on the "Construct Triangle 1" button. This animation bisects angle *BED*, then rotates the resulting triangle 108° to the same orientation as the original triangle. Measure the length of $\overline{CG}$ by clicking on the "Measure Segment CG" button.

5. **What is the ratio of $\frac{BD}{CG}$? $\frac{CG}{BD}$? How do these numbers compare?**

$\frac{BD}{CG} = \frac{12.33}{7.62} \approx 1.618$; $\frac{CG}{BD} = \frac{7.62}{12.33} \approx 0.618$; the ratios are reciprocals of each other.



6. **How does ΔCDG compare to ΔBED? How do you know?**

ΔCDG and ΔBED are similar triangles because their corresponding angles are congruent; by the AA similarity theorem, $\Delta CDG \sim \Delta BED$.

Facilitation Questions

- What does an angle bisector do?
  An angle bisector cuts an angle in half, into two smaller congruent angles.
- What do you need to know to show triangle congruence?
  Corresponding sides are congruent, corresponding angles are congruent. Side-side-side, side-angle-side, angle-side-angle, angle-angle-side are all possible combinations to show triangle congruence.
- What do you need to know to show triangle similarity?
  Corresponding angles are congruent, corresponding sides are proportional.
- Do you see any pairs of congruent angles?
  $\angle BED \cong \angle CDG$, $\angle EBD \cong \angle DCG$, $\angle BDE \cong \angle CGD$

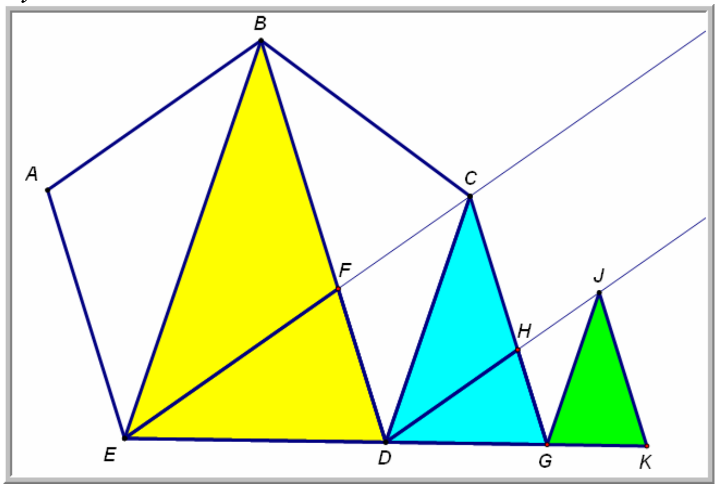
7. **What scale factor could be applied to ΔBED to generate ΔCDG? Have you seen this ratio before? If so, where?**

A scale factor of 0.618 was used. This number is the reciprocal of phi, the golden ratio.

Click the "Construct Triangle 2" button. This animation constructs ΔJGK in the same manner as the construction of ΔCDG. Measure the length of $\overline{JK}$ by clicking the "Measure Segment JK" button.

8. How does ΔJGK compare to ΔCDG? How do you know?

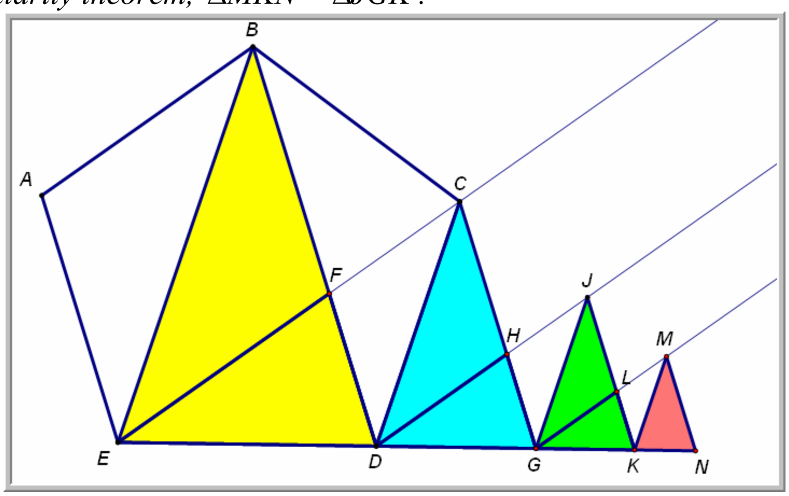
ΔJGK and ΔCDG are similar triangles because their corresponding angles are congruent; by the AA similarity theorem, $\Delta JGK \sim \Delta CDG$.



Click the "Construct Triangle 3" button. This animation constructs ΔMKN in the same manner as the construction of ΔJGK. Measure the length of $\overline{MN}$ by clicking the "Measure Segment MN" button.

9. How does ΔMKN compare to ΔJGK? How do you know?

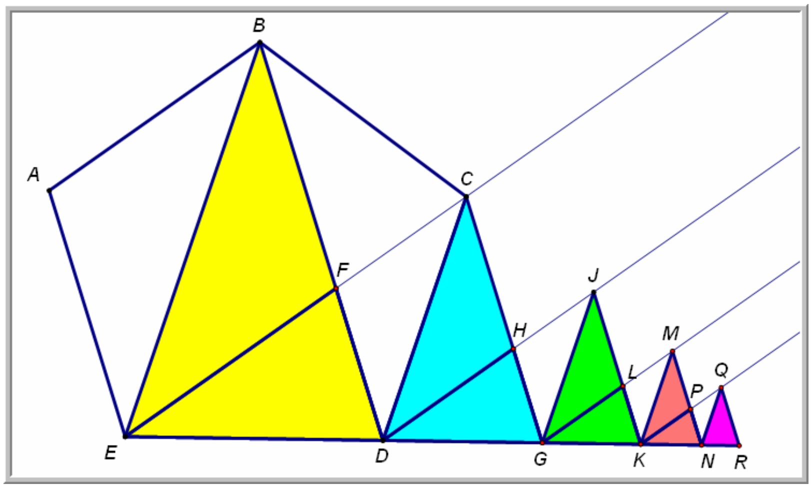
ΔMKN and ΔJGK are similar triangles because their corresponding angles are congruent; by the AA similarity theorem, $\Delta MKN \sim \Delta JGK$.



Click the "Construct Triangle 4" button. This animation constructs ΔQNR in the same manner as the construction of ΔMKN. Measure the length of $\overline{QR}$ by clicking the "Measure Segment QR" button.

10. How does ΔQNR compare to ΔMKN? How do you know?

ΔQNR and ΔMKN are similar triangles because their corresponding angles are congruent; by the AA similarity theorem, $\Delta QNR \sim \Delta MKN$.



11. What patterns do you observe in the sequence of triangles?

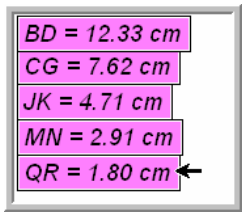
Each triangle is created by taking the angle bisector of the previous triangle and rotating it about a point. Investigation of segment lengths reveals that all of the triangles are similar.

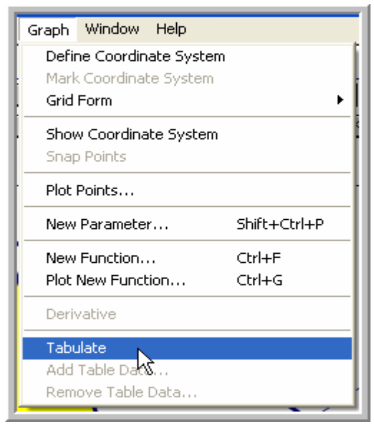
12. Record the measures of the leg of each triangle in the following table.

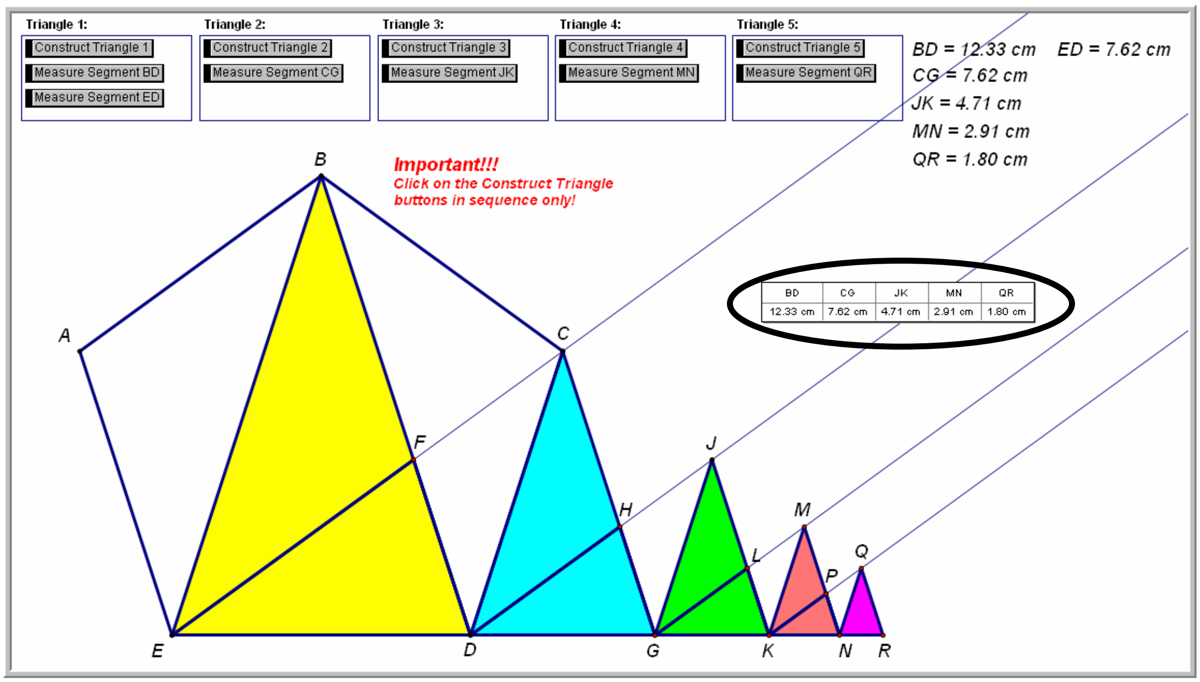
Note: Sample answers appear in the table below. Participants' actual measures will vary depending on the screen resolution and the settings in Geometer's Sketchpad. All of the sample answers that are generated from data that participants will collect are generated from this data set.

Triangle Name	#	Name of Leg	Length of Leg	Process	Ratio
ΔBED	0	BD	12.33		
ΔCDG	1	CG	7.62	$\frac{CG}{BD}=\frac{7.62}{12.33}$	$\frac{CG}{BD}=0.618$
ΔJGK	2	JK	4.71	$\frac{JK}{CG}=\frac{4.71}{7.62}$	$\frac{JK}{CG}=0.618$
ΔMKN	3	MN	2.91	$\frac{MN}{JK}=\frac{2.91}{4.71}$	$\frac{MN}{JK}=0.618$
ΔQNR	4	QR	1.80	$\frac{QR}{MN}=\frac{1.80}{2.91}$	$\frac{QR}{MN}=0.618$

Technology Tip: Once participants have measured the length of each leg, if they select each measurement then choose Tabulate from the Graph menu, Geometer's Sketchpad will create a table as shown in the figure. Then, participants can copy the data into the table readily.

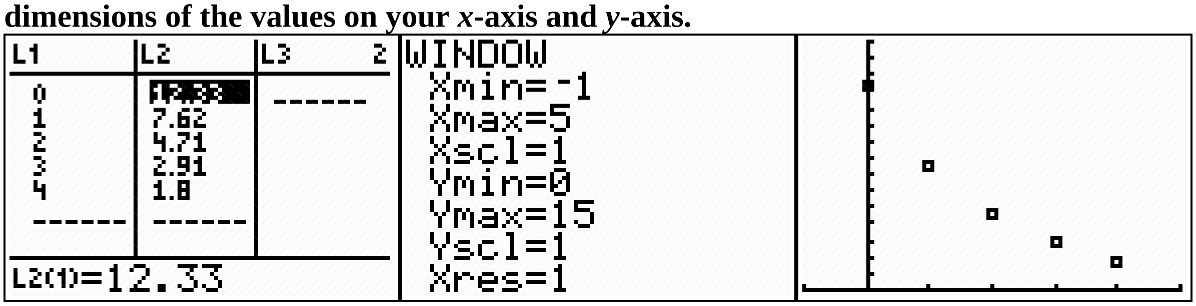






13. Record the ratio of each leg length to its previous leg length in the table.

14. Use an appropriate technology to generate a scatterplot of Leg Length vs. Triangle Number (let ΔBED be Triangle Number 0). Sketch your scatterplot and indicate the dimensions of the values on your *x*-axis and *y*-axis.



15. Based on your scatterplot, what type of function would model the relationship found in the data? Justify your choice.

Answers may vary. Participants should notice a non-linear decreasing curve. An exponential decay model might model the data set well.

Facilitation Questions

- What parameters does your parent function have?
  Answers may vary. Exponential functions are generally of the form $y = ab^x$.
- What do these parameters represent?
  Answers may vary. For exponential functions of the form $y = ab^x$*, a represents the initial value (when* $x = 0$*) and b represents the constant ratio between consecutive y-values.*

16. Use the parent function from Question 15 to determine a function rule to describe the relationship between triangle number and leg length. What do the variables in your function rule represent? What do the constants in your function rule represent?

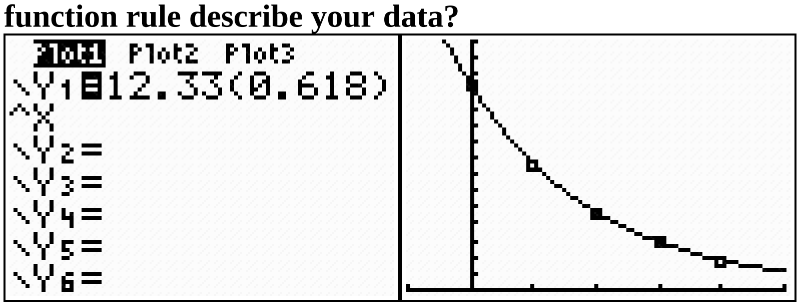
Sample function rule: $y = 12.33\left(0.618\right)^x$

Let x represent the triangle number and y represent the length of the leg of that isosceles triangle. 12.33 represents the leg length of the initial triangle, which was constructed from the pentagon in this geometric sequence. The exponential base, 0.618, is the successive ratio of 0.618 which is the reciprocal of phi, or $\frac{1}{\Phi}$*. The base represents the rate of dilation in this geometric sequence. Since it is less than 1, this function represents an exponential decay.*

Facilitation Questions

- Are the data points increasing or decreasing?
  decreasing
- Are the data points increasing or decreasing at a constant rate?
  No. The rate of decrease slows as the triangle number gets larger.

17. Graph your function rule over the scatterplot. Sketch your graph. How well does the function rule describe your data?



The function rule smoothly connects all five data points indicating a good fit for this data set.

18. Compare the domain of your data and the domain of the function rule.

The domain of the data set is {0, 1, 2, 3, 4} and the domain of the function is all real numbers or $\left\{x : x \in \Re\right\}$*. The domain of the data set is a discrete subset of the domain of the function.*

19. Compare the range of your data and the range of the function rule.
The range of the data set is {1.8, 2.91, 4.71, 7.62, 12.33}. The range of the function is $\{y : y \in \Re, y > 0\}$. *The range of the data set is a discrete subset of the range of the function.*

20. What will be the length of the leg of the 9th triangle in this sequence? Explain how you determined your answer.
For the 9th triangle, x = 9. Use the Table feature of a graphing calculator to find the y-value when x = 9.

X	Y_1
5	1.1115
6	.6869
7	.4245
8	.26234
9	.16213
10	.1002
11	.06192

X=9

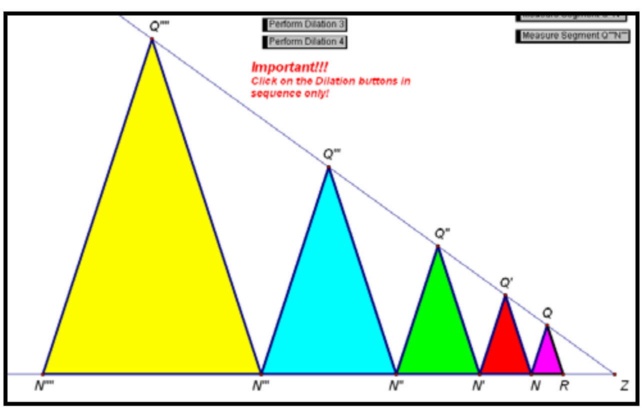
21. Which triangle will be the first one to have a leg length less than 0.5 cm? Explain how you determined your answer.
A leg length of 0.5 cm has a y-value of 0.5 in our function rule. Symbolically, this inequality is $0.5 < 12.23(0.618)^x$. *Use the Table feature of the graphing calculator to find the first x-value that satisfies the inequality. The 7th triangle in the series is the first one to have a leg length of less than 0.5 cm.*

X	Y_1
5	1.1115
6	.6869
7	.4245
8	.26234
9	.16213
10	.1002
11	.06192

X=7

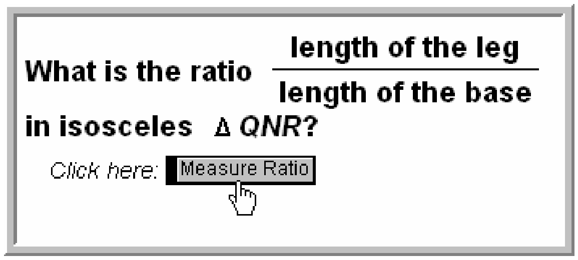
Part 2: Investigating Dilations

In this part, participants will click to a new page in the Geometer's Sketchpad sketch, Golden Triangles.gsp. This time, we will start with ΔQNR, which was the last triangle constructed in the previous sketch. Recall that ΔQNR is a golden isosceles triangle.

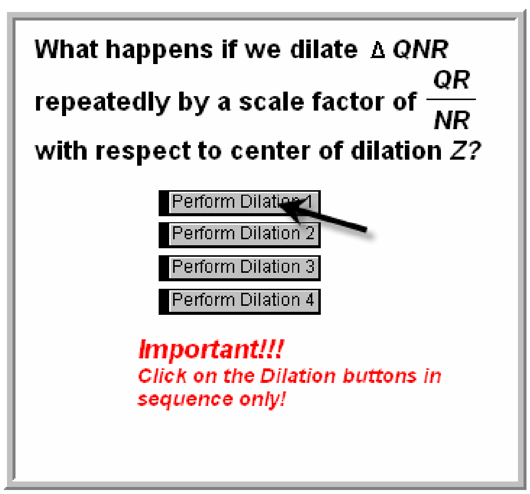


In the previous activity, you constructed a series of golden isosceles triangles. What happens if we take a golden triangle and enlarge it repeatedly?

1. **In the same Geometer's Sketchpad sketch, click on the "Investigating Dilations" tab. In isosceles ΔQNR, what is the ratio of the length of the leg, *QR*, to the length of the base, *NR*?**
   Participants may obtain their answer by clicking the "Measure Ratio" button in the top left corner of the sketch. The ratio is 1.618, which is phi, the golden ratio.



2. **Click the "Perform Dilation 1 button." Describe what you see.**
   $\overrightarrow{ZQ}$ and $\overrightarrow{ZN}$ appear. ΔQNR dilates along these rays by a scale factor of $\frac{QR}{NR} \approx 1.618 = \Phi$ creating $\Delta Q'N'N$.



3. **Click the remaining "Perform Dilation" buttons in sequential order, one at a time. Describe the result.**
   Three more triangles are generated, each one by a scale factor of $\frac{QR}{NR} \approx 1.618 = \Phi$ from the previous triangle.

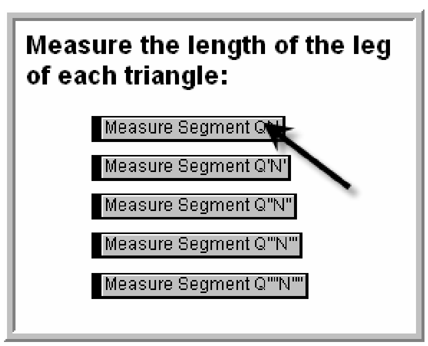
4. How do each of the triangles compare with each other? How do you know?

Each of the five triangles are similar to each other. Their side lengths are all proportional by the same scale factor, $\frac{QR}{NR} \approx 1.618 = \Phi$.

Facilitation Questions

- What geometric properties does a dilation have?
  A dilation is a proportional enlargement or reduction. Thus, the figure is enlarged or reduced in such a way that the side lengths of the image are proportional to the side lengths of the preimage.
- Which corresponding sides are congruent?
  No corresponding sides are congruent. They are proportional by a scale factor of 1.618, or Φ.
- Which corresponding angles are congruent?
  Three sets of corresponding angles are congruent: the set of vertex angles for each isosceles triangle and both sets of base angles.

5. Measure the leg lengths of each triangle by clicking the "Measure Segment" buttons in order, one at a time. Record the segment lengths in the table below.



Note: Sample answers appear in the table below. Participants' actual measures will vary depending on the screen resolution and the settings in Geometer's Sketchpad. This data set generates all of the sample answers generated from data that participants will collect.

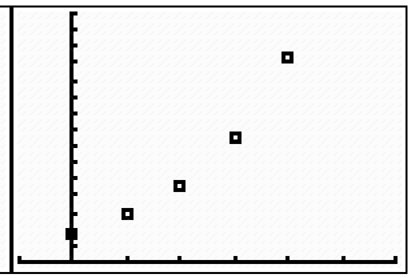
Triangle		Name of Leg	Length of Leg	Process	Ratio
Name	Dilation Number				
ΔQNR	0	QN	1.80		
$\Delta Q'N'N$	1	$Q'N'$	2.91	$\frac{Q'N'}{QN}=\frac{2.91}{1.80}\approx 1.618$	$\frac{Q'N'}{QN}=1.618$
$\Delta Q''N''N'$	2	$Q''N''$	4.71	$\frac{Q''N''}{Q'N'}=\frac{4.71}{2.91}\approx 1.618$	$\frac{Q''N''}{Q'N'}=1.618$
$\Delta Q'''N'''N''$	3	$Q'''N'''$	7.62	$\frac{Q'''N'''}{Q''N''}=\frac{7.62}{4.71}\approx 1.618$	$\frac{Q'''N'''}{Q''N''}=1.618$
$\Delta Q''''N''''N'''$	4	$Q''''N''''$	12.33	$\frac{Q''''N''''}{Q'''N'''}=\frac{12.33}{7.62}\approx 1.618$	$\frac{Q''''N''''}{Q'''N'''}=1.618$

6. **Record the successive ratios in the appropriate column of your table. Use an appropriate technology to generate a scatterplot of Leg Length vs. Dilation Number. Sketch your scatterplot and indicate the dimensions of the values on your *x*-axis and *y*-axis.**

L1	L2	L3 1
0	1.8	------
1	2.91	
2	4.71	
3	7.62	
4	12.33	

L1(6)=

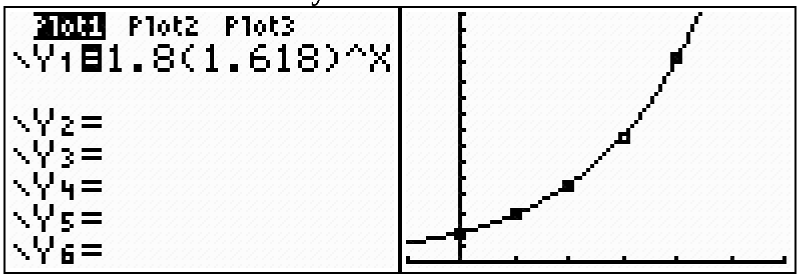
WINDOW
Xmin=-1
Xmax=6
Xscl=1
Ymin=0
Ymax=15
Yscl=1
Xres=1



7. **Based on your scatterplot, what type of function would model the relationship found in the data? Justify your choice.**
Answers may vary. Participants should notice a non-linear increasing curve. An exponential growth model might model the data set well.

8. **Use the parent function from Question 7 to determine a function rule to describe the relationship between dilation number and leg length. What do the variables in your function rule represent? What do the constants in your function rule represent?**
Sample function rule: $y=1.80(1.618)^x$
Let x represent the dilation number and y represent the length (in centimeters) of the leg of that isosceles triangle. 1.80 represents 1.80 cm, the leg length of the initial triangle. The exponential base, 1.618, is the scale factor of dilation which is the golden ratio, Φ.

9. **Graph your function rule over the scatterplot. Sketch your graph. How well does the function rule describe your data?**



The function rule smoothly connects all five data points indicating a good fit for this data set.

10. **What scale factor could be used to generate the second dilation from the original triangle without generating the first dilation?**
Φ^2
The first dilation is generated by a scale factor of Φ. To generate the second dilation, we multiply again by a scale factor of Φ. Multiplying the original dimensions by Φ × Φ is the same as multiplying by Φ^2.

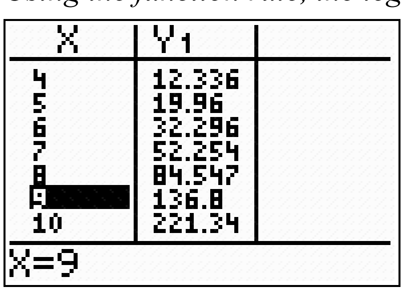
Facilitation Questions

- What does a dilation do to the side lengths of a figure?
  A dilation multiplies the side lengths of a preimage by a scale factor.
- Arithmetically, how can we notate repeated multiplication?
  Using exponents

11. **What scale factor could be used to generate the third dilation from the original triangle without generating the first two dilations?**
Φ^3
Since each successive dilation is created by another multiplication of the dimensions by Φ, a third dilation will multiply the original dimensions by a factor of Φ^3.

12. **How could you predict the scale factor in terms of the dilation number?**
Raise Φ to the power of the dilation number.

13. **What scale factor would be used to generate the 9th dilation in the sequence? What would the leg length of this triangle be? Explain how you determined your answer.**
Φ^9
Using the function rule, the leg length would be $y = 1.80(\Phi)^9 = 1.80(1.618)^9 \approx 136.8\ cm$.



14. Which dilation will be the first one to have a leg length of at least 2.5 meters? Explain how you determined your answer.

A leg length of 2.5 meters has a y-value of 250 cm in our function rule. Symbolically, this inequality is $2.50 \geq 1.80(1.618)^x$ *. Use the Table feature of the graphing calculator to find the first x-value that satisfies the inequality. The 11*th *triangle in the series is the first one to have a leg length of less at least 2.5 meters.*

X	Y1	
6	32.296	
7	52.254	
8	84.547	
9	136.8	
10	221.34	
11	358.13	
12	579.45	

X=11

Explain

This phase of the training should be a whole group discussion. Pose the following questions to participants one at a time, allowing enough time for meaningful discourse to take place about each question.

In this phase, use the debrief questions to prompt participant groups to discuss their responses to the data analysis. At this stage in the professional development, participants should be familiar with using the graphing calculator and a spreadsheet. If none of the participant groups use one of these methods, ask them how they could have used that method to analyze the data. This information is important to the discussion of relative advantages and disadvantages of different types of technology. The reasons that a participant group did not choose a particular technology are as important (if not more so) than the justifications a group gives for the technology that they did choose.

1. **How did you develop your scatterplots? Why did you choose this method?**
   Ask participants to share their methods and their reasons for making that choice. If none of the participants choose one of the technologies (graphing calculator or spreadsheet), ask participants why no one made that choice.

   See "Technology Tutorial: A Golden Idea" for detailed instructions.

2. **How did you develop your function rules? Why did you choose this method?**
   Ask participants to share their methods and their reasons for making that choice. If none of the participants chooses one of the technologies (graphing calculator or spreadsheet), ask participants why no one made that choice.

   See "Technology Tutorial: A Golden Idea" for detailed instructions.

3. **In what ways are the domain and range for the situation and the domain and range for the function rule used to model the situations?**
The domain and range for the situation are each subsets of the domain and range of the function rules, respectively.

4. **How were expressions evaluated in this exploration?**
When given triangle or dilation number, participants were asked to find the leg length.

5. **How were equations solved in this exploration?**
When given a leg length participants were asked to find a triangle or dilation number.

6. **How are the bases of the two exponential functions from Part 1 and Part 2 related?**
In Part 1, the exponential decay function had a base of 0.618, or $\frac{1}{\Phi}$. *In Part 2, the exponential growth function had a base of 1.618, or* Φ. *They are reciprocals of one another.*

Note to Leader: *Record or have a participant volunteer record the responses to Questions 6 and 7 on chart paper to use in the Elaborate phase of the professional development.*

7. **What are the relative advantages and disadvantages of using a graphing calculator to solve this problem?**
Responses may vary.

   The data analysis can be done in a few keystrokes. The power to set your own parameters and graph the function rule empowers the participant to use numerical analysis to calculate meaningful parameters such as a constant of variation. The graphical analysis features of the calculator make it easy to use the graph to solve problems by tracing and calculating the intersection of lines.

   However, the small screen is difficult to see, and the axes in the window cannot be labeled.

8. **What are the relative advantages and disadvantages of using a spreadsheet to solve this problem?**
Responses may vary.

   The regression equation is calculated quickly on the spreadsheet. The axes can be clearly labeled with numbers and text labels. Labeled axes help the participant to use the graph to estimate solutions to problems that can be solved graphically. The graph can be enlarged or reduced then copied and pasted into other computer documents such as a Word or PowerPoint document to communicate the solution to a problem.

   However, the participant is limited to the regression equations available in the spreadsheet. There are no graphical analysis features in most spreadsheets, so only estimates rather than exact solutions can be obtained graphically.

9. **Does the use of technology in this exploration reinforce pencil and paper symbolic algebraic manipulation? If so how? If not, what questions need to be asked so that pencil and paper symbolic algebraic manipulation is reinforced?**
   Answers will vary. The point of this question is to stimulate discourse as to the importance of teacher questioning regardless of the environment that students are working in not to evaluate the use of technology or pencil and paper procedures.

10. **What TEKS does this activity address?**
    Participants should brainstorm a list of TEKS that they believe they have covered in this activity. The Leader Notes contain a comprehensive list of the TEKS addressed in this phase of the professional development. If participants do not mention some of these TEKS, then ask them how the activity also covers them. Since this activity also incorporates Geometry TEKS that are assessed on 11th Grade Exit Level TAKS, this is a good opportunity to discuss the integration of Geometry and Algebra 2 TEKS.

11. **How does the technology that you used enhance the teaching of those TEKS?**
    Responses may vary. However, participants should note that using technology enables them to explore a mathematical concept to a much deeper level. For example, in this activity, using a spreadsheet or the List Editor in a graphing calculator allows participants to make quick computations that allow them to determine the constant multiplier via successive quotients.

    Additionally, the technology of the Geometer's Sketchpad enables participants to quickly and easily generate data in a geometric context that can be used to explore functional relationships. The advantage to using a geometric context is that it affords Algebra 2 teachers an opportunity to review skills that are tested in the geometry objectives on TAKS while teaching Algebra 2 TEKS at the same time.

A Golden Idea: Intentional Use of Data

1. At the close of *A Golden Idea*, distribute the **Intentional Use of Data** activity sheet to each participant.

2. Prompt the participants to work in pairs to identify those TEKS that received greatest emphasis during this activity. Also prompt the participants to identify two key questions that are emphasized during this activity. Allow four minutes for discussion.

Facilitation Questions
- Which mathematics TEKS form the primary focus of this activity?
- What additional mathematics TEKS support the primary TEKS?
- How do these TEKS translate into guiding questions to facilitate student exploration of the content?
- How do your questions reflect the depth and complexity of the TEKS?
- How do your questions support the use of technology?

3. As a whole group, discuss responses for two to three minutes.

4. As a whole group, identify the level(s) of rigor (based on Bloom's taxonomy) addressed, the types of data, the setting, and the data sources used during this Explore/Explain cycle. Allow three minutes for discussion.

Facilitation Question

- What attributes of the activity support the level of rigor that you identified?

5. As a whole group, discuss how this activity might be implemented in other settings. Allow five minutes for discussion.

6. Prompt the participants to set aside the completed Intentional Use of Data activity sheet for later discussion. These completed activity sheets will be used during the Elaborate phase as prompts for generating attributes of judicious users of technology.

Facilitation Questions

- How would this activity change if we had access to one computer (one graphing calculator, CBR, etc.) per participant?
- How would this activity change if we had access to one computer (one graphing calculator, CBR, etc.) per small group of participants?
- How would this activity change if we had access to one computer (one graphing calculator, CBR, etc.) for the entire group of participants?
- How would this activity change if we had used computer-based applications instead of graphing calculators?
- How might we have made additional use of available technologies during this activity?
- How does technology enhance learning?

Sample responses:

TEKS		*a(5), a(6), 2A.1B, 2A.11B, 2A.11F*	
		G.5A, G.10A, G.11A	
Question(s) to Pose to Students	**Math**	*What other situations can you think of that could be modeled with an exponential function?* *What patterns did you discover? What other patterns could there be?*	
	Tech	*How did the dynamic geometry software help you collect data?* *How did the graphing calculator or Excel help you analyze the data?*	
Cognitive Rigor		Knowledge	√
		Understanding	√
		Application	√
		Analysis	√
		Evaluation	√
		Creation	√
Data Source(s)		Real-Time	*Yes; using Geometer's Sketchpad generated on-the-spot data*
		Archival	
		Categorical	
		Numerical	
Setting		Computer Lab	*Each student uses the computer.*
		Mini-Lab	*In groups students take turns or groups switch out.*
		One Computer	*A student operates the control as other students read directions, entire class records data.*
		Graphing Calculator	*Could be used to enter data and find relationships.*
		Measurement-Based Data Collection	*Could be done at stations or individually.*
Bridge to the Classroom		*This activity transfers directly to the classroom with the only modifications being the settings addressed above.*	

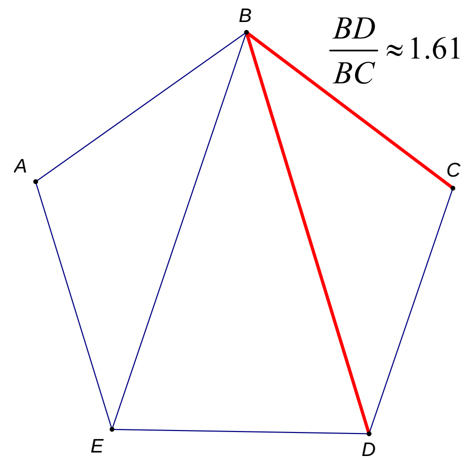
A Golden Idea

Mathematics has been an important influence throughout civilization, from ancient times to the present. In ancient Egypt, Greece, and Rome, geometry and proportion were used in art and architecture. Medieval Europeans carried this tradition of using proportion as they built beautiful cathedrals. Renaissance painters and sculptors used proportion to convey their idea of natural beauty.

Where does this proportion originate? The Greeks found a certain ratio to be prevalent in the natural world around them.

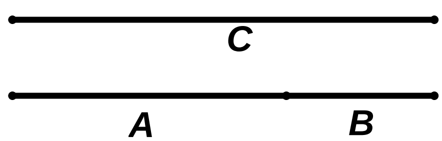
The golden ratio can be developed from several constructions. One way to construct the golden ratio is using the diagonals of a regular pentagon.

In regular pentagon *ABCDE* (shown at right), the ratio of the length of a diagonal from vertex B to the length of a side of the pentagon is always the same. This ratio is called the **golden ratio**, which the Greeks notated with the capital letter *phi*, or Φ.



$$\frac{BD}{BC} \approx 1.618 = \Phi$$

From a segment length perspective, the golden ratio is a geometric mean. Geometrically, segment (in the diagram below, of length C) can be split into two smaller segments (of lengths A and B).

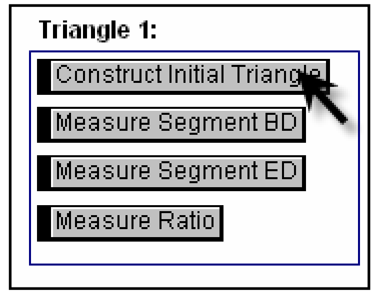


The splitting of the segment is such that the ratio of the length of the original segment to the length of the larger piece (C:A) is the same as the ratio of the length of the larger piece to the length of the smaller piece (A:B). In other words,

$$\frac{C}{A} = \frac{A}{B}$$

If the golden ratio is applied in succession to a geometric construction, what types of functional behaviors are present?

Part 1: Investigating Leg Length

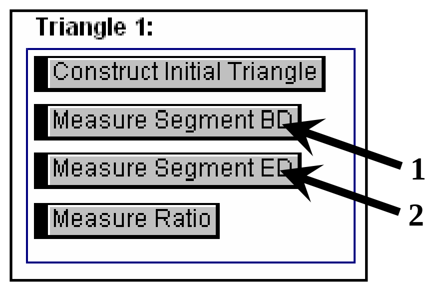


Open the Geometer's Sketchpad sketch "Golden Triangles.GSP." Pentagon *ABCDE* is a regular pentagon. From this regular pentagon, a series of triangles can be constructed. Click on the "Construct Initial Triangle" action button.

1. What kind of triangle is ΔBED? How do you know?

2. Is your triangle classification from Question 1 true for every triangle formed when two diagonals are drawn from one vertex of a regular pentagon? How do you know?

Measure the length of $\overline{BD}$ by clicking on the "Measure Segment BD" action button. Measure the length of $\overline{ED}$ by clicking on the "Measure Segment ED" action button.



3. What is the ratio of the length of $\overline{BD}$ to the length of $\overline{ED}$? How did you find this ratio?

4. What does this ratio represent?

Click on the "Construct Triangle 1" button. This animation bisects angle *BED*, then rotates the resulting triangle 108° to the same orientation as the original triangle. Measure the length of $\overline{CG}$ by clicking on the "Measure Segment CG" button.

5. What is the ratio of $\frac{BD}{CG}$? $\frac{CG}{BD}$? How do these numbers compare?

6. How does ΔCDG compare to ΔBED? How do you know?

7. What scale factor could be applied to ΔBED to generate ΔCDG? Have you seen this ratio before? If so, where?

Click on the "Construct Triangle 2" button. This animation constructs ΔJGK in the same manner as the construction of ΔCDG. Measure the length of $\overline{JK}$ by clicking on the "Measure Segment JK" button.

8. How does ΔJGK compare to ΔCDG? How do you know?

Click the "Construct Triangle 3" button. This animation constructs ΔMKN in the same manner as the construction of ΔJGK. Measure the length of $\overline{MN}$ by clicking the "Measure Segment MN" button.

9. How does ΔMKN compare to ΔJGK? How do you know?

Click the "Construct Triangle 4" button. This animation constructs ΔQNR in the same manner as the construction of ΔMKN. Measure the length of $\overline{QR}$ by clicking the "Measure Segment QR" button.

10. How does ΔQNR compare to ΔMKN? How do you know?

11. What patterns do you observe in the sequence of triangles?

12. Record the measures of the leg of each triangle in the following table.

Triangle		Name of Leg	Length of Leg	Process	Ratio
Name	#				
ΔBED					
ΔCDG					$\frac{CG}{BD} =$
ΔJGK					$\frac{JK}{CG} =$
ΔMKN					$\frac{MN}{JK} =$
ΔQNR					$\frac{QR}{MN} =$

13. Record the ratio of each leg length to its previous leg length in the table.

14. Use an appropriate technology to generate a scatterplot of Leg Length vs. Triangle Number (let ΔBED be Triangle Number 0). Sketch your scatterplot and indicate the dimensions of the values on your *x*-axis and *y*-axis.

15. Based on your scatterplot, what type of function would model the relationship found in the data? Justify your choice.

16. Use the parent function from Question 15 to determine a function rule to describe the relationship between triangle number and leg length. What do the variables in your function rule represent? What do the constants in your function rule represent?

17. Graph your function rule over the scatterplot. Sketch your graph. How well does the function rule describe your data?

18. Compare the domain of your data and the domain of the function rule.

19. Compare the range of your data and the range of the function rule.

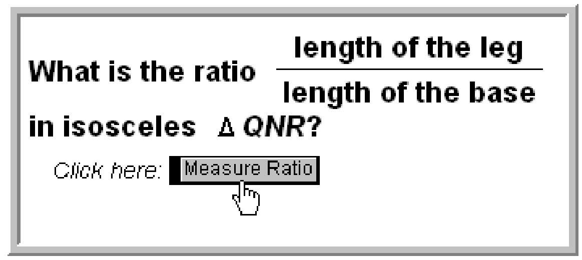
20. What will be the length of the leg of the 9th triangle in this sequence? Explain how you determined your answer.

21. Which triangle will be the first one to have a leg length less than 0.5 cm? Explain how you determined your answer.

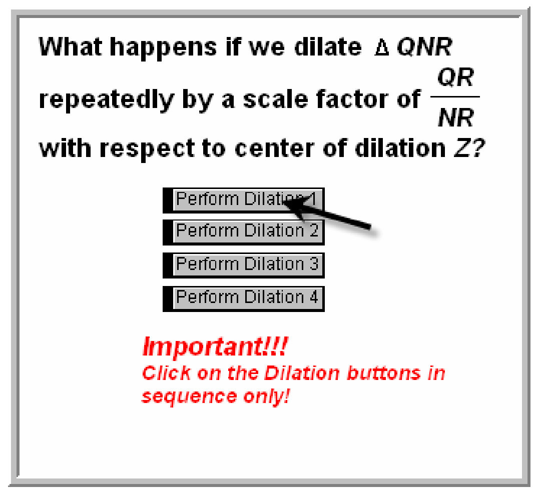
Part 2: Investigating Dilations

In the previous activity, you constructed a series of golden isosceles triangles. What happens if we take a golden triangle and enlarge it repeatedly?

1. In the same Geometer's Sketchpad sketch, click on the "Investigating Dilations" tab. In isosceles ΔQNR, what is the ratio of the length of the leg, QR, to the length of the base, NR?



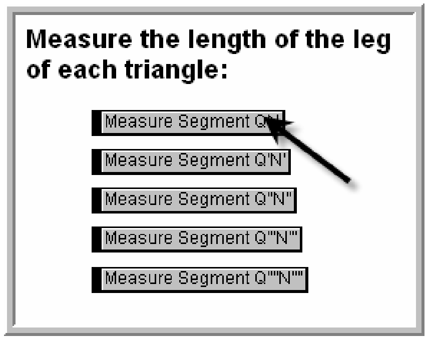
2. Click the "Perform Dilation 1 button." Describe what you see.



3. Click the remaining "Perform Dilation" buttons in sequential order, one at a time. Describe the result.

4. How do each of the triangles compare with each other? How do you know?

5. Measure the leg lengths of each triangle by clicking the “Measure Segment” buttons in order, one at a time. Record the segment lengths in the table below.



Triangle		Name of Leg	Length of Leg	Process	Ratio
Name	Dilation Number				
ΔQNR	*0*				
$\Delta Q'N'N$	*1*				$\frac{Q'N'}{QN} =$
$\Delta Q''N''N'$	*2*				$\frac{Q''N''}{Q'N'} =$
$\Delta Q'''N'''N''$	*3*				$\frac{Q'''N'''}{Q''N''} =$
$\Delta Q''''N''''N'''$	*4*				$\frac{Q''''N''''}{Q'''N'''} =$

6. Record the successive ratios in the appropriate column of your table. Use an appropriate technology to generate a scatterplot of Leg Length vs. Dilation Number. Sketch your scatterplot and indicate the dimensions of the values on your *x*-axis and *y*-axis.

7. Based on your scatterplot, what type of function would model the relationship found in the data? Justify your choice.

8. Use the parent function from Question 7 to determine a function rule to describe the relationship between dilation number and leg length. What do the variables in your function rule represent? What do the constants in your function rule represent?

9. Graph your function rule over the scatterplot. Sketch your graph. How well does the function rule describe your data?

10. What scale factor could be used to generate the second dilation from the original triangle without generating the first dilation?

11. What scale factor could be used to generate the third dilation from the original triangle without generating the first two dilations?

12. How could you predict the scale factor in terms of the dilation number?

13. What scale factor would be used to generate the 9th dilation in the sequence? What would the leg length of this triangle be? Explain how you determined your answer.

14. Which dilation will be the first one to have a leg length of at least 2.5 meters? Explain how you determined your answer.

A Golden Idea: Intentional Use of Data

TEKS		
Question(s) to Pose to Students	Math	
	Tech	
Cognitive Rigor	Knowledge	
	Understanding	
	Application	
	Analysis	
	Evaluation	
	Creation	
Data Source(s)	Real-Time	
	Archival	
	Categorical	
	Numerical	
Setting	Computer Lab	
	Mini-Lab	
	One Computer	
	Graphing Calculator	
	Measurement-Based Data Collection	
Bridge to the Classroom		